Biometrical Letters
Vol. 46 (2009), No. 2, 103-112

On some constructions of regular D-optimal spring balance weighing designs

Bronisław Ceranka[1], Małgorzata Graczyk[1], Krystyna Katulska[2]

[1]Poznań University of Life Sciences, Department of Mathematical and Statistical Methods, Wojska Polskiego 28, 60-637 Poznań, Poland,
e-mail: bronicer@up.poznan.pl, magra@up.poznan.pl
[2]Adam Mickiewicz University, Faculty of Mathematics and Computer Science, Umultowska 87, 61-614 Poznań, Poland, e-mail: krakat@amu.edu.pl

Dedicated to Professor Tadeusz Caliński for his 80th birthday

Summary

The paper deals with the problem of constructions of spring balance weighing designs satisfying the criterion of D-optimality. The incidence matrices of balanced incomplete block designs and partially balanced incomplete block designs are used in constructions of regular D-optimal spring balance weighing designs.

Key words: balanced incomplete block design, D-optimal design, partially balanced incomplete block design.

1. Introduction

The results of n weighing operations determining the individual weights of p objects fit into the linear model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}, \tag{1}$$

where $\mathbf{y}$ is the $n \times 1$ random vector of observations, $\mathbf{X} = (x_{ij})$, $i = 1, 2, ..., n$, $j = 1, 2, ..., p$, is a $n \times p$ matrix of known elements with $x_{ij} = 1$ or 0 according as in the ith weighing operation the jth object is placed on the pan or not, $\mathbf{w}$ is the $p \times 1$ vector of unknown weights of objects and $\mathbf{e}$ is an $n \times 1$ random vector of errors. We assume that there are no systematic errors, i.e. $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and the variances of errors are not equal and the errors are not correlated, i.e. $\mathrm{Var}(\mathbf{e}) = \sigma^2\mathbf{G}$, where $\mathbf{0}_n$ denotes the $n \times 1$

vector with zero elements everywhere, and $\mathbf{G}$ is the known $n \times n$ diagonal positive definite matrix.

For the estimation of individual unknown weights of objects we use the normal equations

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{w} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{y}.$$

A spring balance weighing design is said to be singular or nonsingular, according as the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is singular or nonsingular respectively. It is clear that if $\mathbf{G}$ is a known positive definite diagonal matrix then the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular if and only if the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular, i.e. if and only if $\mathbf{X}$ is of full column rank $\mathrm{r}(\mathbf{X}) = p$.

However, if $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, then the generalized least squares estimator of $\mathbf{w}$ is given by

$$\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y} \tag{2}$$

and the variance matrix of $\hat{\mathbf{w}}$ is

$$\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}. \tag{3}$$

Various aspects of spring balance weighing designs have been studied by, for example Raghavarao (1971) and Banerjee (1975).

In many problems concerning optimum weighing experiments D-optimal designs are considered. There are designs for which the determinant of $(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}$ is minimal, i.e. the determinant of $(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})$ is maximal. The conditions determining the existence of D-optimal weighing designs under the assumption that the errors are uncorrelated and have the same variances were studied by Gail and Kiefer (1980, 1982), and Neubauer et al. (1998).

In this paper constructions are given for the regular D-optimal spring balance weighing designs based on theorems stated in Neubauer et al. (1998) and Katulska and Przybył (2007) for some special forms of $\mathbf{G}$ and $\mathbf{X}$.

2. Regular D-optimal designs

For given n and p let us consider the $n \times p$ design matrix $\mathbf{X}$ in the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{x}' \end{bmatrix} \tag{4}$$

or in the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{x}' \\ \mathbf{z}' \end{bmatrix}, \tag{5}$$

where $\mathbf{x}$ and $\mathbf{z}$ are $p \times 1$ vectors of elements 1 or 0, $\mathbf{X}_1$ is $h \times p$ design matrix which satisfies the conditions given by Neubauer et al. (1998)

$$\mathbf{X}_1'\mathbf{X}_1 = \frac{(p+1)h}{4p}\left(\mathbf{I}_p + \mathbf{1}_p\mathbf{1}_p'\right) \quad \text{if } p \text{ is odd} \tag{6}$$

and

$$\mathbf{X}_1'\mathbf{X}_1 = \frac{(p+2)h}{4(p+1)}\left(\mathbf{I}_p + \mathbf{1}_p\mathbf{1}_p'\right) \quad \text{if } p \text{ is even,} \tag{7}$$

where $\mathbf{I}_p$ is the $p \times p$ identity matrix and $\mathbf{1}_p$ denotes the $p \times 1$ vector of ones. Let us note that $h = n - 1$ or $h = n - 2$ if $\mathbf{X}$ is given in the form (4) or (5) respectively.

The definitions and theorems below are from Katulska and Przybył (2007).

Definition 2.1. Any nonsingular spring balance weighing design with the design matrix $\mathbf{X}$ given in (4) and with the diagonal variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is in the form

$$\mathbf{G} = \operatorname{diag}(1, ..., 1, g^{-1}), \quad g > 0 \tag{8}$$

is said to be regular D-optimal if

$$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) = \begin{cases} (p+1)\left(\frac{(p+1)(n-1)}{4p}\right)^p\left(1 + \frac{gp}{n-1}\right) & \text{if } p \text{ is odd} \\ (p+1)\left(\frac{(p+2)(n-1)}{4(p+1)}\right)^p\left(1 + \frac{gp}{n-1}\right) & \text{if } p \text{ is even} \end{cases}.$$

Definition 2.2. Any nonsingular spring balance weighing design with the design matrix $\mathbf{X}$ given in (5) and with the variance matrix of errors $\sigma^2\mathbf{G}$, where

$$\mathbf{G} = \operatorname{diag}(1, ..., 1, g_1^{-1}, g_2^{-1}), \quad g_1, g_2 > 0 \tag{9}$$

is said to be regular D-optimal if

(i) for odd p

$$\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right) = \begin{cases} \eta\left(\beta + g_1g_2p^2\right) & \text{if } (p+1) \equiv 0(\text{mod}4) \\ \eta\left(\beta + g_1g_2\frac{p^2(p-1)(p+3)}{(p+1)^2}\right) & \text{if } (p+3) \equiv 0(\text{mod}4) \end{cases}$$

(ii) for even p

$$\det\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)=\begin{cases}\xi\left(\beta+g_1g_2\frac{p^2(p+1)(p+3)}{(p+2)^2}\right) & \text{if } p\equiv 0(\text{mod}4)\\ \xi\left(\beta+g_1g_2(p^2-1)\right) & \text{if } (p+2)\equiv 0(\text{mod}4)\end{cases}$$

where $\eta=\frac{p+1}{(n-2)^2}\left(\frac{(p+1)(n-2)}{4p}\right)^p$, $\beta=(n-2)^2+(g_1+g_2)p(n-2)$, $\xi=\frac{p+1}{(n-2)^2}\left(\frac{(p+2)(n-2)}{4(p+1)}\right)^p$.

Theorem 2.1. Any nonsingular spring balance weighing design with the design matrix $\mathbf{X}$ in the form (4) and with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in (8), is regular D-optimal

(i) for odd p, if condition (6) is fulfilled and $\mathbf{x}^{'}\mathbf{1}_p=\frac{p+1}{2}$,

(ii) for even p, if condition (7) is fulfilled and $\mathbf{x}^{'}\mathbf{1}_p=\frac{p}{2}$ or $\mathbf{x}^{'}\mathbf{1}_p=\frac{p+2}{2}$.

Theorem 2.2. Any nonsingular spring balance weighing design with the design matrix $\mathbf{X}$ in the form (5) and with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in (9), is regular D-optimal

(i) for odd p, if condition (6) is fulfilled and moreover $\mathbf{x}^{'}\mathbf{1}_p=\mathbf{z}^{'}\mathbf{1}_p=\frac{p+1}{2}$ and

$$\mathbf{x}^{'}\mathbf{z}=\begin{cases}\frac{p+1}{4} & \text{if } (p+1)\equiv 0(\text{mod}4)\\ \frac{p-1}{4}\text{ or }\frac{p+3}{4} & \text{if } (p+3)\equiv 0(\text{mod}4)\end{cases},$$

(ii) for even p, if condition (7) is fulfilled and moreover

a) $\mathbf{x}^{'}\mathbf{1}_p=\mathbf{z}^{'}\mathbf{1}_p=\frac{p}{2}$ and

$$\mathbf{x}^{'}\mathbf{z}=\begin{cases}\frac{p}{4} & \text{if } p\equiv 0(\text{mod}4)\\ \frac{p-2}{4} & \text{if } (p+2)\equiv 0(\text{mod}4)\end{cases},$$

b) $\mathbf{x}^{'}\mathbf{1}_p=\frac{p}{2}$, $\mathbf{z}^{'}\mathbf{1}_p=\frac{p+2}{2}$ or $\mathbf{x}^{'}\mathbf{1}_p=\frac{p+2}{2}$, $\mathbf{z}^{'}\mathbf{1}_p=\frac{p}{2}$ and

$$\mathbf{x}^{'}\mathbf{z}=\begin{cases}\frac{p}{4} & \text{if } p\equiv 0(\text{mod}4)\\ \frac{p+2}{4} & \text{if } (p+2)\equiv 0(\text{mod}4)\end{cases},$$

c) $\mathbf{x}^{'}\mathbf{1}_p=\mathbf{z}^{'}\mathbf{1}_p=\frac{p+2}{2}$ and

$$\mathbf{x}^{'}\mathbf{z}=\begin{cases}\frac{p}{4}+1 & \text{if } p\equiv 0(\text{mod}4)\\ \frac{p+2}{4} & \text{if } (p+2)\equiv 0(\text{mod}4)\end{cases}.$$

3. Constructions of the design matrices

We define the balanced incomplete block design and the partially balanced incomplete block design with two associate classes (see, for example, Raghavarao and Padgett, 2005).

A balanced incomplete block design is an arrangement of v treatments in b blocks, each of size k, in such a way that each treatment occurs at most once in each block, occurs in exactly r blocks and every pair of treatments occurs together in exactly λ blocks. The integers v, b, r, k, λ are called parameters of the balanced incomplete block design. Let $\mathbf{N}$ be the incidence matrix of this design. It is straightforward to verify that $vr = bk,\ \lambda(v-1) = r(k-1),\ \mathbf{NN}' = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$.

A partially balanced incomplete block design with two associate classes is an arrangement of v treatments in b blocks, each of size k such that every treatment occurs at most once in a block and occurs in r blocks. Each treatment has exactly n_q qth associates, $q = 1, 2$. Two treatments which are qth associate occur together in exactly λ_q blocks. The numbers $v,\ b,\ r,\ k,\ \lambda_q,\ q = 1, 2$ are parameters of the partially balanced incomplete block design. The partially balanced incomplete block design is usually identified by the association scheme of treatments.

A group divisible design is a partially balanced incomplete block design with two associate classes for which the $v = ms$ treatments may be divided into m groups of s distinct treatments each, such that treatments belonging to the same group are first associates and two treatments belonging to different groups are second associates, $n_1 = s-1$, $n_2 = s(m-1)$, $(s-1)\lambda_1 + (m-1)\lambda_2 = r(k-1)$.

Let us consider the design matrix $\mathbf{X}$ in the form (4) or (5) with $\mathbf{X}_1 = \mathbf{N}'$, where $\mathbf{N}$ is the $v \times b$ incidence matrix of the balanced incomplete block design with parameters $v,\ b,\ r,\ k,\ \lambda$. Thus $p = v$ and $h = b$. Hence from (6) we have

$$\mathbf{X}_1'\mathbf{X}_1 = \mathbf{NN}' = \frac{(v+1)b}{4v}\left(\mathbf{I}_v + \mathbf{1}_v\mathbf{1}_v'\right). \tag{10}$$

On the other hand, $\mathbf{NN}' = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$. Thus (10) is satisfied if and only if $r = 2\lambda$. Taking into consideration the equality $\frac{r}{2} = \frac{(v+1)b}{4v}$ we obtain $v = 2k-1$ therefore we have

Theorem 3.1. If there exists an incidence matrix $\mathbf{N}$ of the balanced incomplete block design with parameters $v = 2k-1$, $b = \frac{2\lambda(2k-1)}{k}$, $r = 2\lambda$,

k, λ, then $\mathbf{X}$ given in (4) or (5) with $\mathbf{X}_1 = \mathbf{N}'$ and the variance matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (8) or (9), respectively, is a regular D-optimal spring balance weighing design.

Based on Raghavarao (1971) and Ceranka and Katulska (1994), we have:

Theorem 3.2. Let $\mathbf{N}$ be the incidence matrix of the balanced incomplete block design with parameters

(i) $v = b = 4t + 3,\ r = k = 2(t + 1),\ \lambda = t + 1,$ where $4t + 3$ is a prime or a prime power,

(ii) $v = 4t + 1,\ b = 2(4t + 1),\ r = 2(2t + 1),\ k = \lambda = 2t + 1,$ where $4t + 1$ is a prime or a prime power,

(iii) $v = t^2,\ b = 2t^2,\ r = t^2 + 1,\ k = \lambda = \frac{1}{2}(t^2 + 1),$ where t^2 is a prime power, $t \neq 2$.

If $\mathbf{X}_1 = \mathbf{N}'$ then

(1) if $n = b + 1$ then $\mathbf{X}$ in the form (4) with the variance matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (8)

(2) if $n = b + 2$ then $\mathbf{X}$ in the form (5) with the variance matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (9)

is the design matrix of a regular D-optimal spring balance weighing design.

Proof. It is easy to check that the parameters given in (i)-(iii) satisfy the conditions given in Theorem 3.1. □

Neubauer et al. (1998) revolved the regular D-optimal spring balance weighing designs for the case $n = b$. For $v = 4t + 3$ they considered the incidence matrices of the balanced incomplete block designs constructed from $4t \times 4t$ Hadamard matrices. The incidence matrices of complementary designs to these balanced incomplete block designs are the regular spring balance weighing designs. The parameters of these complementary designs are of the form (i) of Theorem 3.2. In the same paper the case $v = 4t + 1$ is also considered. The incidence matrices of the balanced incomplete block designs for this case are constructed from a supplementary difference set in the Galois field. The authors also considered incidence matrices of complementary designs to such designs and proved that these designs are regular D-optimal spring balance weighing designs. The parameters of these complementary designs are of the form (ii) of Theorem 3.2.

Theorem 3.3. If v is even, then a regular D-optimal spring balance weighing design $\mathbf{X}$ in the form (4) or (5) with $\mathbf{X}_1 = \mathbf{N}'$ and the variance matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (8) or (9), respectively, does not exist.

Proof. For the design matrix $\mathbf{X}$ in the form (4) or (5) we have $\mathbf{X}_1'\mathbf{X}_1 = \mathbf{N}\mathbf{N}' = \frac{(v+1)b}{4v}\left(\mathbf{I}_v + \mathbf{1}_v\mathbf{1}_v'\right)$. Similarly as in the proof of Theorem 3.1 we obtain $\mathbf{X}_1'\mathbf{X}_1 = \lambda\left(\mathbf{I}_v + \mathbf{1}_v\mathbf{1}_v'\right)$. However the balance incomplete block design for which $\frac{r}{2} = \frac{(v+2)b}{4(v+1)}$ does not exist. □

Now, based on the incidence matrices of two group divisible designs with the same association scheme, we construct the regular D-optimal spring balance weighing design for even p.

We consider the $n \times p$ design matrix $\mathbf{X}$ in the form (4) or (5) with $\mathbf{X}_1 = [\mathbf{N}_1 \quad \mathbf{N}_2]'$, where $\mathbf{N}_i$ is the incidence matrix of the group divisible design with the same association scheme with parameters $v,\ b_i,\ r_i,\ k_i,\ \lambda_{1i},\ \lambda_{2i}$, $i = 1, 2$, and let

$$\lambda_{11} + \lambda_{12} = \lambda_{21} + \lambda_{22} = \lambda. \tag{11}$$

Theorem 3.4. Let p be even. If there exist incidence matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ of the group divisible design with the same association scheme with parameters $v,\ b_i,\ r_i,\ k_i,\ \lambda_{1i},\ \lambda_{2i},\ i = 1, 2$, and

(i) $r_1 + r_2 = 2\lambda$

(ii) $b_1 + b_2 = \frac{2(v+1)(r_1+r_2)}{v+2} = \frac{4\lambda(v+1)}{v+2}$

then $\mathbf{X}$ given in (4) or (5) with $\mathbf{X}_1 = [\mathbf{N}_1 \quad \mathbf{N}_2]'$ and the variance matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (8) or (9), respectively, is a regular D-optimal spring balance weighing design.

Proof. Let $p = v$ and $h = b_1 + b_2$. From condition (7) we have

$$\mathbf{X}_1'\mathbf{X}_1 = \mathbf{N}_1\mathbf{N}_1' + \mathbf{N}_2\mathbf{N}_2' = \frac{(v+2)(b_1+b_2)}{4(v+1)}\left(\mathbf{I}_v + \mathbf{1}_v\mathbf{1}_v'\right). \tag{12}$$

On the other hand, $\mathbf{N}_1\mathbf{N}_1' + \mathbf{N}_2\mathbf{N}_2' = (r_1 + r_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$. Thus (12) is satisfied if and only if $r_1 + r_2 = 2\lambda$. Taking into consideration the equality $\frac{(v+2)(b_1+b_2)}{4(v+1)} = \frac{r_1+r_2}{2}$ we obtain the condition (ii). Hence the result. □

Theorem 3.5. Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be the incidence matrices of group divisible designs with the same association scheme and (11) be satisfied with parameters

(1) $v = 6$ and

(1.1) $b_1 = 3$, $r_1 = 2$, $k_1 = 4$, $\lambda_{11} = 2$, $\lambda_{21} = 1$ and $b_2 = 4$, $r_2 = 2$, $k_2 = 3$, $\lambda_{12} = 0$, $\lambda_{22} = 1$,

(1.2) $b_1 = 6$, $r_1 = k_1 = \lambda_{11} = 4$, $\lambda_{21} = 2$ and $b_2 = 8$, $r_2 = 4$, $k_2 = 3$, $\lambda_{12} = 0$, $\lambda_{22} = 2$,

(1.3) $b_1 = 9$, $r_1 = 6$, $k_1 = 4$, $\lambda_{11} = 6$, $\lambda_{21} = 3$ and $b_2 = 12$, $r_2 = 6$, $k_2 = 3$, $\lambda_{12} = 0$, $\lambda_{22} = 3$,

(1.4) $b_1 = 9$, $r_1 = 6$, $k_1 = 4$, $\lambda_{11} = 3$, $\lambda_{21} = 4$ and $b_2 = 12$, $r_2 = 6$, $k_2 = 3$, $\lambda_{12} = 3$, $\lambda_{22} = 2$,

(1.5) $b_1 = 12$, $r_1 = 8$, $k_1 = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 4$ and $b_2 = 16$, $r_2 = 8$, $k_2 = 3$, $\lambda_{12} = 0$, $\lambda_{22} = 4$,

(1.6) $b_1 = 15$, $r_1 = 10$, $k_1 = 4$, $\lambda_{11} = 10$, $\lambda_{21} = 5$ and $b_2 = 20$, $r_2 = 10$, $k_2 = 3$, $\lambda_{12} = 0$, $\lambda_{22} = 5$,

(2) $v = 10$ and $b_1 = 10$, $r_1 = k_1 = \lambda_{11} = 6$, $\lambda_{21} = 3$ and $b_2 = 12$, $r_2 = 6$, $k_2 = 5$, $\lambda_{11} = 0$, $\lambda_{22} = 3$,

(3) $v = 14$ and

(3.1) $b_1 = 14$, $r_1 = k_1 = \lambda_{11} = 8$, $\lambda_{21} = 4$ and $b_2 = 16$, $r_2 = 8$, $k_2 = 7$, $\lambda_{12} = 0$, $\lambda_{22} = 4$,

(3.2) $b_1 = 7$, $r_1 = 4$, $k_1 = 8$, $\lambda_{11} = 4$, $\lambda_{21} = 2$ and $b_2 = 8$, $r_2 = 4$, $k_2 = 7$, $\lambda_{12} = 0$, $\lambda_{22} = 2$,

(4) $v = 18$ and $b_1 = 18$, $r_1 = k_1 = \lambda_{11} = 10$, $\lambda_{21} = 5$ and $b_2 = 20$, $r_2 = 10$, $k_2 = 9$, $\lambda_{11} = 0$, $\lambda_{22} = 5$.

If $\mathbf{X}_1 = [\mathbf{N}_1 \quad \mathbf{N}_2]'$ then

(i) if $n = b_1 + b_2 + 1$ then $\mathbf{X}$ in the form (4) with the variance matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (8)

(ii) if $n = b_1 + b_2 + 2$ then $\mathbf{X}$ in the form (5) with the variance matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (9)

is the design matrix of a regular D-optimal spring balance weighing design.

Proof. It is easy to check that the parameters given in (1)-(4) satisfy conditions (i) and (ii) of Theorem 3.4. □

4. Examples

Example 4.1. Let $n = 11$ and $p = 5$. Then $b = 10 = n - 1$ and there exists the balance incomplete block design given in Theorem 3.2 (ii) with parameters $v = 5,\ b = 10,\ r = 6,\ k = 3,\ \lambda = 3$ given by the incidence matrix

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Hence if $\mathbf{X}_1 = \mathbf{N}'$ and $\mathbf{x}' = [1\ 1\ 1\ 0\ 0]$ then

$$\mathbf{X}' = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

is the design matrix of a regular D-optimal spring balance weighing design with the variance matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (8).

Example 4.2. Let $n = 9$ and $p = 6$. Then $b_1 + b_2 = 7 = n - 2$ and there exist the group divisible block designs with the same association scheme given in Theorem 3.5 (1.1.) with parameters $v = 6,\ b_1 = 3,\ r_1 = 2,\ k_1 = 4,\ \lambda_{11} = 2,\ \lambda_{21} = 1$ and $v = 6,\ b_2 = 4,\ r_2 = 2,\ k_2 = 3,\ \lambda_{12} = 0,\ \lambda_{22} = 1$ given by the incidence matrices $\mathbf{N}_1,\ \mathbf{N}_2$ and association scheme, where

$$\mathbf{N}_1' = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{N}_2' = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \qquad \begin{matrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{matrix},$$

for $m = 3,\ s = 2$ (see Clatworthy, 1973). For a given treatment, the first associate is a treatment in the same row ($n_1 = 1$) whereas the remaining treatments are the second associates ($n_2 = 4$). Hence if $\mathbf{X}_1 = [\mathbf{N}_1\ \mathbf{N}_2]'$,

$\mathbf{x}' = [1\ 1\ 1\ 0\ 0\ 0]$ and $\mathbf{z}' = [0\ 0\ 1\ 1\ 1\ 0]$ then

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

is the design matrix of a regular D-optimal spring balance weighing design with the variance matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (9).

References

Banerjee K.S. (1975): Weighing Designs for Chemistry, Medicine, Economics, Operations Research, Statistics. Marcel Dekker Inc., New York.

Ceranka B., Katulska K. (1994): Constructions of the optimum chemical balance weighing design with non-homogeneity of the variances of errors. Advances in Statistical Software 4: 189–196 Editor: F. Faulbaum, Stuttgard-Jena-New York: Gustav Fisher.

Clatworthy W.H. (1973): Tables of two-associate-class partially balanced designs. NBS Applied Mathematics Series 63.

Gail Z., Kiefer J. (1980): D-optimum weighing designs. Ann. Statistics 8: 1293–1306.

Gail Z., Kiefer J. (1982): Construction methods D-optimum weighing designs when $n \equiv 3(\text{mod}4)$. Ann. Statistics 10: 502–510.

Katulska K., Przybył K. (2007): On certain D-optimal spring balance weighing designs. Journal of Statistical Theory and Practice 1: 393–404.

Neubauer M.G., Watkins W., Zeitlin J. (1998): Notes on D-optimal designs. Linear Algebra and its Applications 280: 109–127.

Raghavarao D. (1971): Constructions and Combinatorial Problems in Design of Experiments. John Wiley Inc. New York.

Raghavarao D., Padgett L.V. (2005): Block designs. Analysis, Combinatorics and Applications. Series of Applied Mathematics 17, Word Scientific Publishing Co. Pte. Ltd.